

# Two-loop fermion self-energy and propagator in reduced QED<sub>3,2</sub>

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We compute the two-loop fermion self-energy in massless reduced quantum electrodynamics (RQED) for an arbitrary gauge in the case where the photon field is three-dimensional and the fermion field two-dimensional: super-renormalizable RQED<sub>3,2</sub> with  $N_F$  fermions. We find that the theory is infrared finite at two-loop and that finite corrections to the fermion propagator have a remarkably simple form.

One of the building blocks of multi-loop calculations is the two-loop massless propagator diagram, see Fig. 1:

$$\int \int \frac{d^{d_e} k_1 d^{d_e} k_2}{[-(k_1 + p)^2]^{\alpha_1} [-(k_2 + p)^2]^{\alpha_2} [-k_2^2]^{\alpha_3} [-k_1^2]^{\alpha_4} [-(k_2 - k_1)^2]^{\alpha_5}} = - \frac{\pi^{d_e}}{(-p^2)^{\sum_{i=1}^5 \alpha_i - d_e}} G(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5), \quad (1)$$

where  $G$  is the so-called coefficient function of the diagram,  $\alpha_i$  are arbitrary indices and  $p$  is an external momentum in a Minkowski space-time of dimensionality  $d_e$ . This diagram is at the heart of numerous radiative correction calculations in quantum field theory and associated to the development of sophisticated methods such as, *e.g.*, the Gegenbauer polynomial technique<sup>1,2</sup>, integration by parts<sup>3,4</sup>, and the method of uniqueness<sup>3,5,6</sup>, see Ref. [8] for a historical review on this diagram. In the case where all indices are integers, this diagram is well known and can be expressed in terms of recursively one-loop diagrams. When all indices are arbitrary, the result is highly non-trivial and can be represented<sup>9</sup> as a combination of two-fold series. In some intermediate cases, simpler forms can be obtained.<sup>2,3,7,10–12,14</sup> In particular, in Ref. [2], an ingenious transformation was found from Gegenbauer two-fold series to one-fold  ${}_3F_2$ -hypergeometric series of unit argument for a complicated class of diagrams having two integer indices on adjacent lines and three other arbitrary indices. For this class of diagrams, similar results have been found in Ref. [11] using an ansatz to solve the recurrence relations arising from integration by parts. In Ref. [2], the results were applied to the computation of a diagram with a single non-integer index on the central line. This important diagram appears in various calculations, see, *e.g.*, Refs. [3,7,10,13,14]; it was shown in Ref. [2] to reduce to a single  ${}_3F_2$ -hypergeometric series of unit argument. More recently, in Ref. [15], the results of [2] were applied to the case involving two arbitrary indices on non adjacent lines. In this case, the corresponding coefficient function:

$$G(\alpha, 1, \beta, 1, 1) = C_D \left[ \begin{array}{c} \alpha \quad 1 \\ \bullet \quad \bullet \\ | \quad | \\ \bullet \quad \bullet \\ 1 \quad \beta \end{array} \right], \quad (2)$$

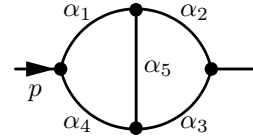


FIG. 1: Two-loop massless propagator diagram.

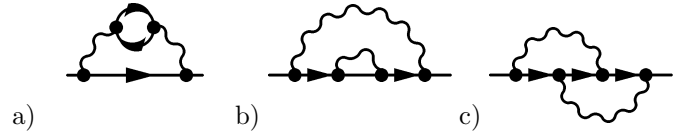


FIG. 2: Two-loop fermion self-energy diagrams.

was shown to reduce to two  ${}_3F_2$ -hypergeometric series of argument 1.

In Ref. [15], Eq. (2) appeared in the computation of the two-loop fermion self-energy in reduced quantum electrodynamics (RQED), [16], or RQED <sub>$d_\gamma, d_e$</sub> , see also Refs. [17] in relation with RQED<sub>4,3</sub>. In the general case, this relativistic model describes the interaction of an abelian  $U(1)$  gauge field living in  $d_\gamma$  space-time dimensions with a fermion field localized in a reduced space-time of  $d_e$  dimensions ( $d_e \leq d_\gamma$ ). In RQED <sub>$d_\gamma, d_e$</sub> , while the bubble and rainbow diagrams, Figs. 2a) and 2b), respectively, naturally reduce to recursively one-loop diagrams, the crossed photon diagram, Fig. 2c), involves a contribution of the type Eq. (2) with the indices given by:

$$\alpha = \beta = 1 - \varepsilon_e, \quad (3)$$

where, following the notation of Ref. [15],  $d_\gamma = 4 - 2\varepsilon_\gamma$  and  $d_e = 4 - 2\varepsilon_e - 2\varepsilon_\gamma$ . In the case of usual QEDs, *e.g.*, QED<sub>4</sub> and QED<sub>3</sub>, the parameter  $\varepsilon_e = 0$  and all indices are integers. Reduced models appear to be more com-

plicated, *a priori*, as they generally involve non-integer indices, *e.g.*,  $\varepsilon_e = 1/2$  and  $\varepsilon_\gamma \rightarrow 0$  for RQED<sub>4,3</sub> which corresponds to the ultrarelativistic limit of an undoped graphene monolayer. This complication turned out to be overcome, in the case of RQED<sub>4,d<sub>e</sub></sub>, by the presence of a coefficient  $\varepsilon_\gamma \rightarrow 0$  in factor of the ultra-violet (UV) convergent Eq. (2) in the expression of the self-energy.

In this Brief Report we complete the previous study by examining the case of RQED<sub>3,2</sub> which is interesting from the field theory point of view as it does require the com-

putation of Eq. (2) for  $\varepsilon_e = 1/2$  and  $\delta_\gamma = \varepsilon_\gamma - 1/2 \rightarrow 0$ . We shall show that the formulas of Ref. [15] are extremely convenient to perform such a task. Moreover, similarly to QED<sub>3</sub> ( $\varepsilon_e = 0$  and  $\delta_\gamma \rightarrow 0$ ), RQED<sub>3,2</sub> is super-renormalizable and therefore asymptotically free. However, contrary to QED<sub>3</sub> where infra-red (IR) divergences yield an anomalous dimension to the fermion field at two loop, RQED<sub>3,2</sub> is finite at two-loop and the corrections to the fermion propagator take a very simple form, as will be shown below.

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Following [15], we start by considering the fermion self-energy up to two loops in RQED<sub>3,d<sub>e</sub></sub>:

$$\Sigma_V(p^2) = \frac{\tilde{\alpha}}{4\pi} e^{(\gamma_E - L_p)\delta_\gamma} \sigma_1(\varepsilon_e, \delta_\gamma, a) + \left(\frac{\tilde{\alpha}}{4\pi}\right)^2 e^{2(\gamma_E - L_p)\delta_\gamma} \sigma_2(\varepsilon_e, \delta_\gamma, a), \quad (4)$$

where  $L_p = \ln(-p^2/\mu^2)$ ,  $\tilde{\alpha}$  is a momentum-dependent dimensionless coupling constant ( $e^2$  has dimension of mass in RQED<sub>3,d<sub>e</sub></sub>) defined as:  $\tilde{\alpha} = e^2/\sqrt{-4\pi p^2}$  and we have used the fact that charge does not renormalize in RQED<sub>3,d<sub>e</sub></sub>. The one-loop contribution reads:

$$\sigma_1 = \Gamma(1 - \varepsilon_e) \frac{1 - 2\varepsilon_e - 2\delta_\gamma}{2} \left( \frac{\varepsilon_e}{1 - \varepsilon_e - 2\delta_\gamma} - a \right) G(1, 1 - \varepsilon_e), \quad (5)$$

where  $a$  is a gauge fixing parameter ( $a = 1$  in the Feynman gauge) that we shall keep arbitrary in what follows and  $G(\alpha, \beta)$  is the coefficient function of the one-loop massless propagator diagram:

$$G(\alpha, \beta) = \frac{a(\alpha)a(\beta)}{a(\alpha + \beta - d_e/2)} \quad a(\alpha) = \frac{\Gamma(d_e/2 - \alpha)}{\Gamma(\alpha)}. \quad (6)$$

The two-loop function corresponds to the sum of the three diagrams in Fig. 2,  $\sigma_2 = \sigma_a^{(2)} + \sigma_b^{(2)} + \sigma_c^{(2)}$ , where the first two diagrams yield ( $N_F$  is the number of massless fermion fields)

$$\sigma_a^{(2)} = -2N_F \Gamma^2(1 - \varepsilon_e) \frac{(1 - 2\varepsilon_e - 2\delta_\gamma)^2}{3 - 2\varepsilon_e + 2\delta_\gamma} G(1, 1)G(1, 1/2 - \varepsilon_e + \delta_\gamma), \quad (7a)$$

$$\sigma_b^{(2)} = 2\Gamma^2(1 - \varepsilon_e) \frac{\delta_\gamma(1 - 2\varepsilon_e - 2\delta_\gamma)[\varepsilon_e - a(1 - \varepsilon_e - 2\delta_\gamma)]^2}{(1 + 2\delta_\gamma)(1 - \varepsilon_e - 2\delta_\gamma)} G(1, 1 - \varepsilon_e)G(1 - \varepsilon_e, 1/2 + \delta_\gamma), \quad (7b)$$

and the third diagram can be further separated into three parts  $\sigma_c^{(2)} = \sigma_{c_1}^{(2)} + \sigma_{c_2}^{(2)} + \sigma_{c_3}^{(2)}$  where

$$\sigma_{c_1}^{(2)} = \Gamma^2(1 - \varepsilon_e) \frac{1 - 2\varepsilon_e - 2\delta_\gamma}{2} G^2(1, 1 - \varepsilon_e) \left[ 1 + 2\varepsilon_e + 2\delta_\gamma + (1 - a) \frac{(1 - 2\varepsilon_e - 2\delta_\gamma)^2}{1 - \varepsilon_e - 2\delta_\gamma} - \frac{(1 - a)^2}{2} (1 - 2\varepsilon_e - 2\delta_\gamma) \right. \\ \left. + 2 \frac{(\varepsilon_e + 2\delta_\gamma)[8 - (3 - 2\delta_\gamma)(1 + 2\varepsilon_e + 2\delta_\gamma)]}{(1 + 2\varepsilon_e + 6\delta_\gamma)(1 - 2\varepsilon_e - 6\delta_\gamma)} + \frac{4\varepsilon_e}{1 - \varepsilon_e - 2\delta_\gamma} - \frac{2\varepsilon_e}{1 - 2\varepsilon_e - 6\delta_\gamma} \left( 5 + 2\varepsilon_e + 2\delta_\gamma - 4 \frac{\varepsilon_e + 2\delta_\gamma}{1 - \varepsilon_e - 2\delta_\gamma} \right) \right], \quad (8a)$$

$$\sigma_{c_2}^{(2)} = -\Gamma^2(1 - \varepsilon_e) \frac{1 - 2\varepsilon_e - 2\delta_\gamma}{2} G(1, 1 - \varepsilon_e) G(1 - \varepsilon_e, 1/2 + \delta_\gamma) \left[ 2 - 4\varepsilon_e - 4\delta_\gamma + 4 \frac{1 - 2\varepsilon_e - 2\delta_\gamma}{1 - \varepsilon_e - 2\delta_\gamma} - 16 \frac{1 - \delta_\gamma}{1 + 2\delta_\gamma} \right. \\ \left. - 4(1 - a) \frac{\delta_\gamma(1 - 2\varepsilon_e - 2\delta_\gamma)}{1 - \varepsilon_e - 2\delta_\gamma} + 2(1 - a)^2 \delta_\gamma + 2 \frac{\varepsilon_e(5 + 2\varepsilon_e + 2\delta_\gamma)}{\varepsilon_e + 2\delta_\gamma} + 4 \frac{\varepsilon_e(1 - 2\delta_\gamma)}{(1 + 2\delta_\gamma)(1 - \varepsilon_e - 2\delta_\gamma)} \right], \quad (8b)$$

$$\sigma_{c_3}^{(2)} = \Gamma^2(1 - \varepsilon_e) \frac{1 - 2\varepsilon_e - 2\delta_\gamma}{2} G(1 - \varepsilon_e, 1, 1 - \varepsilon_e, 1, 1) \frac{(1 + 2\delta_\gamma)[8 - (3 - 2\delta_\gamma)(1 + 2\varepsilon_e + 2\delta_\gamma)]}{(1 - 2\varepsilon_e - 6\delta_\gamma)(1 + 2\varepsilon_e + 6\delta_\gamma)}. \quad (8c)$$

Straightforward application of the above equations to RQED<sub>3,2</sub> ( $\varepsilon_e = 1/2$  and  $\delta_\gamma \rightarrow 0$ ) yields the following expansions:

$$e^{(\gamma_E - L_p)\delta_\gamma} \sigma_1 = \sqrt{\pi} (1 - a) + \sqrt{\pi} \delta_\gamma [4 - (1 - a)\bar{L}_p] + O(\delta_\gamma^2), \quad (9a)$$

$$e^{2(\gamma_E - L_p)\delta_\gamma} \sigma_a^{(2)} = -4\pi N_F + 4\pi N_F \delta_\gamma (1 + 2\bar{L}_p - 8 \ln 2) + O(\delta_\gamma^2), \quad (9b)$$

$$e^{2(\gamma_E - L_p)\delta_\gamma} \sigma_b^{(2)} = \pi(1 - a)^2 + 2\pi(1 - a) \delta_\gamma [1 + 3a - (1 - a)(\bar{L}_p - 6 \ln 2)] + O(\delta_\gamma^2), \quad (9c)$$

$$e^{2(\gamma_E - L_p)\delta_\gamma} \sigma_{c_1}^{(2)} = -\frac{\pi}{6\delta_\gamma^2} - \frac{\pi(3 - \bar{L}_p)}{3\delta_\gamma} + \pi \left( 3 - (1-a)^2 + \frac{5}{6}\zeta_2 + 2\bar{L}_p - \frac{1}{3}\bar{L}_p^2 \right) \\ + \pi \delta_\gamma \left( \frac{47}{3} + 5\zeta_2 + \frac{55}{9}\zeta_3 - 4\bar{L}_p - \frac{5}{3}\zeta_2\bar{L}_p - 2\bar{L}_p^2 + \frac{2}{9}\bar{L}_p^3 + 8a + 2a^2\bar{L}_p - 4a\bar{L}_p \right) + O(\delta_\gamma^2), \quad (10a)$$

$$e^{2(\gamma_E - L_p)\delta_\gamma} \sigma_{c_2}^{(2)} = \pi(8 - (1-a)^2) + 2\pi\delta_\gamma \left( 4a - 16 - (8 - (1-a)^2)(\bar{L}_p - 6\ln 2) \right) + O(\delta_\gamma^2), \quad (10b)$$

$$e^{2(\gamma_E - L_p)\delta_\gamma} \sigma_{c_3}^{(2)} = \frac{\pi}{6} \frac{(1 + 2\delta_\gamma)[4 - (3 - 2\delta_\gamma)(1 + \delta_\gamma)]}{1 + 3\delta_\gamma} e^{2(\gamma_E - L_p)\delta_\gamma} G(1/2, 1, 1/2, 1, 1), \quad (10c)$$

where  $\bar{L}_p = L_p + 4\ln 2$ ,  $\sigma_{c_1}^{(2)}$  is explicitly IR singular and the last term contains a contribution from the complicated diagram, Eq. (2), which cannot be reduced to products of one-loop massless functions.

Among the various forms derived for  $G(\alpha, 1, \beta, 1, 1)$  in Ref. [15], the most convenient one for the present application is Eq. (B10) in that paper. Together with (B11) and in the case of RQED<sub>3,2</sub>, these equations yield:

$$G(1/2, 1, 1/2, 1, 1) = -24 \frac{\delta_\gamma(1 + 3\delta_\gamma)}{1 + 2\delta_\gamma} \frac{\Gamma^2(1/2 - \delta_\gamma)\Gamma(1 - \delta_\gamma)\Gamma(1 + 2\delta_\gamma)}{\Gamma(1/2)\Gamma(1 - 2\delta_\gamma)\Gamma(1 - 3\delta_\gamma)} I(1/2), \quad (11a)$$

$$I(1/2) = \frac{\Gamma(1/2)}{\Gamma(2 + 2\delta_\gamma)} \frac{\pi \sin[\pi\delta_\gamma]}{\sin[\pi(1/2 + 2\delta_\gamma)]\sin[\pi(1/2 - \delta_\gamma)]} + \sum_{n=0}^{\infty} \frac{\Gamma(n - 2\delta_\gamma)\Gamma(n + 1)}{n!\Gamma(n + 3/2)} \frac{1}{n - 1/2 - 2\delta_\gamma} \\ + \frac{1 + 4\delta_\gamma}{4\delta_\gamma} \frac{\Gamma(1/2 + \delta_\gamma)\Gamma(1 - \delta_\gamma)}{\Gamma(1/2 - 2\delta_\gamma)\Gamma(1 + 2\delta_\gamma)} \frac{\sin[\pi(1/2 + 2\delta_\gamma)]}{\sin[\pi(1/2 - \delta_\gamma)]} \sum_{n=0}^{\infty} \frac{\Gamma(n - 2\delta_\gamma)\Gamma(n - 1 - 3\delta_\gamma)}{n!\Gamma(n - 1/2 - 3\delta_\gamma)} \frac{1}{n - 1/2 - 2\delta_\gamma}. \quad (11b)$$

Indeed, under this form, the  $\delta_\gamma$ -expansion of the hypergeometric functions with non-integer parameters is most easily done. We shall carry such expansion up to  $O(1)$  which is what is needed for  $I$  in order to expand the  $G$ -function up to  $O(\delta_\gamma)$ . The first term in Eq. (11b) is of  $O(\delta_\gamma)$  and can be neglected. The second term is singular and expands as:

$$\sum_{n=0}^{\infty} \frac{\Gamma(n - 2\delta_\gamma)\Gamma(n + 1)}{n!\Gamma(n + 3/2)} \frac{1}{n - 1/2 - 2\delta_\gamma} = \frac{\Gamma(1 - 2\delta_\gamma)}{\Gamma(3/2)} \left[ \frac{1}{\delta_\gamma} + 4G - 6 + O(\delta_\gamma) \right], \quad (12)$$

where  $G$  is Catalan's constant. The third term is conveniently split into two parts following the property that

$$\frac{1}{(n - 1 - 3\delta_\gamma)} \frac{(n - 1/2 - 3\delta_\gamma)}{(n - 1/2 - 2\delta_\gamma)} = \frac{1}{(1 + 2\delta_\gamma)} \left[ \frac{1}{(n - 1 - 3\delta_\gamma)} + \frac{2\delta_\gamma}{n - 1/2 - 2\delta_\gamma} \right]. \quad (13)$$

The first term in the r.h.s of Eq. (13) can be summed exactly as a  ${}_2F_1$ -series of unit argument and is singular:

$$\sum_{n=0}^{\infty} \frac{\Gamma(n - 2\delta_\gamma)\Gamma(n - 1 - 3\delta_\gamma)}{n!\Gamma(n + 1/2 - 3\delta_\gamma)} = \frac{\Gamma(-2\delta_\gamma)\Gamma(-1 - 3\delta_\gamma)\Gamma(3/2 + 2\delta_\gamma)}{\Gamma(1/2 - \delta_\gamma)\Gamma(3/2)} = -\frac{1 + 4\delta_\gamma}{6\delta_\gamma^2(1 + 3\delta_\gamma)} \frac{\Gamma(1 - 2\delta_\gamma)\Gamma(1 - 3\delta_\gamma)\Gamma(1/2 + 2\delta_\gamma)}{\Gamma(1/2 - \delta_\gamma)\Gamma(1/2)} \quad (14)$$

The second term in the r.h.s. of Eq. (13) comes with a factor of  $2\delta_\gamma$ . It is singular and, similarly to (12), the singular part is only in the  $n = 0$  term of the series. So, we have

$$\sum_{n=0}^{\infty} \frac{\Gamma(n - 2\delta_\gamma)\Gamma(n - 3\delta_\gamma)}{n!\Gamma(n + 1/2 - 3\delta_\gamma)} \frac{1}{n - 1/2 - 2\delta_\gamma} = \frac{\Gamma(1 - 2\delta_\gamma)\Gamma(1 - 3\delta_\gamma)}{\Gamma(1/2 - 3\delta_\gamma)} \left[ -\frac{1}{3\delta_\gamma^2} + \frac{4}{3\delta_\gamma} - \frac{16}{3} + 16G - 6\zeta_2 + O(\delta_\gamma) \right]. \quad (15)$$

With the help of Eqs. (14) and (15), we obtain

$$\sum_{n=0}^{\infty} \frac{\Gamma(n - 2\delta_\gamma)\Gamma(n - 1 - 3\delta_\gamma)}{n!\Gamma(n - 1/2 - 3\delta_\gamma)} \frac{1}{n - 1/2 - 2\delta_\gamma} = \frac{\Gamma(1 - 2\delta_\gamma)\Gamma(1 - 3\delta_\gamma)}{\Gamma(1/2 - 3\delta_\gamma)} \left\{ -\frac{1}{6\delta_\gamma^2} - \frac{1}{2\delta_\gamma} + \frac{25}{6} - 3\zeta_2 \right. \\ \left. + \delta_\gamma \left[ -\frac{41}{2} + 32G - 9\zeta_2 + 7\zeta_3 \right] + O(\delta_\gamma^2) \right\}. \quad (16)$$

Combining all terms up to  $O(\delta_\gamma)$  yields:

$$I(1/2) = \frac{\Gamma(1-2\delta_\gamma)}{24\sqrt{\pi}} \left\{ -\frac{1}{\delta_\gamma^3} - \frac{7-12\ln 2}{\delta_\gamma^2} + \frac{1}{\delta_\gamma} \left( 61 + 6\zeta_2 + 84\ln 2 - 72\ln^2 2 \right) \right. \\ \left. - 311 + 42\zeta_2 + 30\zeta_3 + 384G - 156\ln 2 - 72\zeta_2\ln 2 - 504\ln^2 2 + 288\ln^3 2 + O(\delta_\gamma) \right\}. \quad (17)$$

Therefore:

$$G(1/2, 1, 1/2, 1, 1) = e^{-2\gamma_E\delta_\gamma} \left( \frac{1}{\delta_\gamma^2} + \frac{8(1-\ln 2)}{\delta_\gamma} - 56 - 5\zeta_2 - 64\ln 2 + 32\ln^2 2 \right. \\ \left. + \delta_\gamma \left( 240 - 40\zeta_2 - \frac{110}{3}\zeta_3 - 384G - 128\ln 2 + 40\zeta_2\ln 2 + 256\ln^2 2 - \frac{256}{3}\ln^3 2 \right) + O(\delta_\gamma^2) \right). \quad (18)$$

Substituting the result of Eq. (18) in Eq. (10c), yields:

$$e^{2(\gamma_E-L_p)\delta_\gamma} \sigma_{c_3}^{(2)} = \frac{\pi}{6\delta_\gamma^2} + \frac{\pi(3-\bar{L}_p)}{3\delta_\gamma} + \pi \left( -11 - \frac{5}{2}\zeta_2 - 2\bar{L}_p + \frac{1}{3}\bar{L}_p^2 \right) \\ + \pi \delta_\gamma \left( \frac{193}{3} - 5\zeta_2 - \frac{55}{9}\zeta_3 - 64G - 96\ln 2 + 22\bar{L}_p + \frac{5}{3}\zeta_2\bar{L}_p + 2\bar{L}_p^2 - \frac{2}{9}\bar{L}_p^3 \right) + O(\delta_\gamma^2). \quad (19)$$

All divergent terms cancel each-other in the crossed-photon diagram which therefore turns out to be IR finite:

$$e^{2(\gamma_E-L_p)\delta_\gamma} \sigma_c^{(2)} = -2\pi(1-a)^2 + 4\pi\delta_\gamma \left( 12 + 4a - 16G + (1-a)^2(\bar{L}_p - 3\ln 2) \right) + O(\delta_\gamma^2). \quad (20)$$

The total two-loop self-energy then reduces to:

$$e^{2(\gamma_E-L_p)\delta_\gamma} \sigma_2 = -4\pi N_F - \pi(1-a)^2 + 2\pi\delta_\gamma \left( 2N_F(1+2\bar{L}_p - 8\ln 2) + 25 + 10a - 3a^2 - 32G + (1-a)^2\bar{L}_p \right) + O(\delta_\gamma^2). \quad (21)$$

The theory is therefore finite ( $Z_\psi = 1$ ) and the expression of the dressed fermion propagator reads:

$$-i\not{p} S(p) = 1 + \frac{\tilde{\alpha}}{4\pi} \sqrt{\pi} \left( 1 - a + \delta_\gamma(4 - (1-a)\bar{L}_p) + O(\delta_\gamma^2) \right) \\ + \left( \frac{\tilde{\alpha}}{4\pi} \right)^2 \left( -4\pi N_F + 4\pi\delta_\gamma \left( N_F(1+2\bar{L}_p - 8\ln 2) + 16(1-G) - \frac{3}{2}(1-a)^2 \right) + O(\delta_\gamma^2) \right) + O(\tilde{\alpha}^3). \quad (22)$$

Remarkably, the  $O(1)$  two-loop correction is gauge-invariant and reduces to a very simple form:  $-4\pi N_F$  while the  $O(\delta_\gamma)$  correction involves  $\pi$ ,  $\ln 2$  as well as the Clausen function  $\text{Cl}_2(\pi/2) = G$ .

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